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# Laplace–Runge–Lenz-like new constant in many-body systems from post-Newtonian dynamics

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**Abstract**

It has been shown in the past that the classical Laplace–Runge–Lenz vector emerges naturally in the computation of the post-Newtonian Lorentz boost in 2-body electrical or gravitational systems. This procedure is extended here to many-body systems. A new *N*-body vector observable of the Runge–Lenz type is found which is an integral of motion for non-trivial families of solutions. Conditions for its integrability are discussed with explicit examples. The relation of this vector with the post-Newtonian centre-of-mass is also briefly discussed.

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## 1. Introduction

The existence of the so-called Laplace–Runge–Lenz vector[1] as an internal[2] integral of the motion in Newtonian 2-body Kepler/Coulomb systems has been well known for more than two centuries [1]. With the potential $U(r) = \kappa/r$ and the internal angular momentum $\vec{\ell} = \vec{r} \times \vec{p}$, a common definition of the Runge–Lenz vector is

$$\vec{K} = \frac{1}{\mu}\vec{p} \times \vec{\ell} + \frac{\kappa}{r}\vec{r}, \tag{1}$$

with $\mu$ being the reduced mass, $\vec{r}$ the relative coordinate and $\vec{p}$ the corresponding momentum. Knowledge of the Runge–Lenz vector amounts to having a full solution for the configuration of the system: in classical (non-quantum) systems $\vec{K}$ is perpendicular to $\vec{\ell}$, directed along the major axis of the (reduced) particle's trajectory and its magnitude is proportional to the

[1] According to Goldstein's historical review [1], this vector should actually be named after the predecessors of Runge and Lenz, but for the sake of simplicity the common name will be used in the following.

[2] The term *internal* refers here and in the following to any dynamical quantity which depends only on the relative coordinates of the particles and is invariant under uniform global translations.

eccentricity of the curve; in simple quantum systems the Runge–Lenz vector provides a very elegant means for obtaining the full quantum picture of the system (as in the case of the hydrogen atom).

It is well known that the Runge–Lenz vector (1) generates, together with the internal angular momentum $\vec{\ell}$, $SO(4)$ or $SO(3,1)$ symmetry groups which contain the internal rotational symmetry as a subgroup. These symmetries have been shown to exist, by general arguments, for general rotationally symmetric systems with arbitrary number of degrees of freedom [2–4]. Thus, the internal dynamics of rotationally symmetric systems is governed not only by the rotational symmetry but rather by the larger symmetry generated together by (at least) both the internal angular momentum and the Runge–Lenz vector.

In practice, however, we know how to construct Runge–Lenz vectors only for 2-body systems. This presents the challenge of generalizing the Runge–Lenz symmetry also for systems with more than two particles. The search for Runge–Lenz-like vectors in many-body systems then becomes part of the search for integrals of the motion (other than energy, linear and angular momenta) in these systems, which has been unsuccessful in most cases. Of course, Runge–Lenz-like vectors may be found in interaction-free systems. There may also exist examples such as the exactly soluble model of D and R M Lynden-Bell [5] in which they explicitly build a Runge–Lenz vector (actually, *N* different vectors, corresponding to the *N* particles), but that is not the case in realistic systems. Two centuries of study of celestial mechanics [6–8, 10, 11] led to colloquial agreement that the only simple (i.e. analytically expressed in terms of the dynamical variables of the particles) observables are the global quantities of total energy, total momentum and total angular momentum (see also Gutzwiller [12]). In particular cases, such as circular restricted 3-body systems[3], there is also a scalar integral of the motion known as the Jacobi integral [7, 8, 10] which is a kind of energy integral. However, in general 3-body systems the theorem of Bruns [7] holds which states that *all the algebraic integrals of the motion that exist are necessarily algebraic functions of the energy and total linear and angular momenta*. In particular, nowhere is a vector constant of the motion like the Runge–Lenz vector found which corresponds—in a symmetric manner—to all the particles.

So far, all this is well known. It may be therefore somewhat surprising that relativistic considerations lead, in a very elegant way, to integrals of motion of Runge–Lenz type in many-body non-relativistic systems, including celestial mechanics and Coulomb interactions. It has been shown by Dahl [13], apparently unnoticed by the physics community, that the classical Runge–Lenz vector naturally appears—as an integral of motion—in the computation of the Lorentz boost in the post-Newtonian approximation of an electromagnetic or gravitational 2-body system. Dahl's procedure was applied so far only to 2-body systems for which, as Dahl has shown (see also [14]), it leads to an internal constant vector which is proportional to the classical Runge–Lenz vector (1). No attempt has been made, to the author's best knowledge, to apply it to larger systems. Therefore, if Dahl's procedure works also for $N(N \geqslant 3)$-body systems, then a new constant of the motion, unrealized so far, is obtained.

Dahl's procedure consists in splitting the post-Newtonian Lorentz boost in the centre-of-mass (CM) frame into a sum of two terms, one of which contains a vector $\vec{R}_o$ which formally looks like the Newtonian centre-of-mass (see equation (2)). This vector, which in the post-Newtonian approximation is not a constant of the motion, is separated from the boost in such a way that it carries all the translational properties of the boost, so that the remainder is an internal quantity. Since the non-constancy of $\vec{R}_o$ is purely a relativistic effect, its time-varying

[3] Restricted 3-body systems are, in celestial mechanics, systems in which two bodies, known as 'primaries', are much more massive than the third, so their motion is assumed to be on a 2-body unperturbed closed orbit while the third body moves in the common gravitational field of the primaries [6–8].

part is a vector observable of order $1/c^2$. This time-varying part is also internal, since the translational properties of $\vec{R}_o$ are all contained in an arbitrary constant of integration. In this way, an internal vector is separated from the boost which is an integral of the motion yet of order $1/c^2$.

Appreciating the fact that more than two centuries of research yielded a null result regarding the existence of extra algebraically simple integrals of motion in the general $N$-body Kepler/Coulomb problem, Dahl's procedure is not expected to be applicable to all such systems. However, there may exist non-trivial families of solutions for which the procedure works, and an extra vector integral of the motion does exist.

This is indeed what we show in the following: applying Dahl's procedure to Kepler/Coulomb systems with arbitrary number of bodies an explicit expression for such a vector is derived, and the conditions for it being an integral of motion are found. It is an $N$-body Runge–Lenz-like vector, with a kinetic part that is first order in the coordinates and second order in the velocities or momenta, just like the classical Runge–Lenz vector (1). It is demonstrated that it is indeed a non-trivial integral for $N$-body collinear central configurations (in which the ratios between interparticle distances remain unchanged) or for 3-body triangular central configurations, and it is explicitly computed in these cases. Also discussed are the integrability conditions for general 3-body systems. The relation of this vector with the post-Newtonian centre-of-mass is also briefly discussed.

## 2. Dahl's procedure in many-body systems

Consider an $N$-body system with masses $\{m_a\}$, possible electrical charges $\{e_a\}$, coordinates $\{\vec{x}_a\}$ and linear momenta $\{\vec{p}_a\}$ $(a = 1, \dots, N)$, with total Newtonian mass $M_o = \sum_a m_a$ and total linear momentum $\vec{P} = \sum_a \vec{p}_a$ (in the post-Newtonian approximation).

As outlined in the introduction, Dahl's procedure starts with isolating, in the CM reference frame, a term containing the Newtonian CM from the post-Newtonian Lorentz boost. The Newtonian CM is

$$\vec{R}_o = \frac{1}{M_o} \sum_a m_a \vec{x}_a, \tag{2}$$

and the CM frame is defined by $\vec{P} = 0$ without fixing the origin. The post-Newtonian Lorentz boost is [15]

$$\vec{N} = \sum_a \left( m_a + \frac{m_a v_a^2}{2c^2} + \sum_{b \neq a} \frac{\kappa_{ab}}{2 r_{ab} c^2} \right) \vec{x}_a, \tag{3}$$

with $\kappa_{ab} = e_a e_b$ or $\kappa_{ab} = -G m_a m_b$ for electromagnetic or gravitational systems, respectively. Then, with $\vec{r}_a = \vec{x}_a - \vec{R}_o$ being the particles' coordinates relative to $\vec{R}_o$ and $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$ the relative coordinates between particles, the boost (3) becomes

$$\vec{N} = M R_o + \frac{1}{2c^2} \sum_a \left( m_a v_a^2 + \sum_{b \neq a} \frac{\kappa_{ab}}{r_{ab}} \right) \vec{r}_a, \tag{4}$$

where

$$M = \sum_a \left( m_a + \frac{m_a v_a^2}{2c^2} \right) + \sum_{(a,b)} \frac{\kappa_{ab}}{r_{ab} c^2} \tag{5}$$

is the invariant total relativistic mass in the post-Newtonian approximation. $(a, b)$ implies summation over all pairs of different particles. Since the sum in equation (4) is an internal quantity, all the translational properties of $\vec{N}$ (i.e. its behaviour under uniform spatial translations) are contained in $\vec{R}_o$.

The next step is to compute the time-varying part of $\vec{R}_o$ which is of order $1/c^2$. Substituting the post-Newtonian velocities [15]

$$\vec{v}_a = \frac{\vec{p}_a}{m_a} - \frac{p_a^2}{2c^2m_a^3}\vec{p}_a - \sum_{b\neq a}\frac{\kappa_{ab}}{2m_am_bc^2r_{ab}}\left[\vec{p}_b + \frac{(\vec{p}_b\cdot\vec{r}_{ab})}{r_{ab}^2}\vec{r}_{ab} + \alpha m_b\left(\frac{\vec{p}_b}{m_b} - \frac{\vec{p}_a}{m_a}\right)\right] \tag{6}$$

with $\alpha = 0$ or $\alpha = 6$ for the electromagnetic or gravitational case, respectively, the time derivative of $\vec{R}_o$ is found to be, after some algebra,

$$\begin{aligned}
\frac{\mathrm{d}\vec{R}_o}{\mathrm{d}t} &= \sum_a \frac{m_a}{M_o}\vec{v}_a = \frac{1}{M_o}\sum_a\left\{\vec{p}_a - \frac{p_a^2}{2c^2m_a^2}\vec{p}_a\right.\\
&\quad\left. - \sum_{b\neq a}\frac{\kappa_{ab}}{2r_{ab}m_bc^2}\left[\vec{p}_b + \frac{(\vec{p}_b\cdot\vec{r}_{ab})}{r_{ab}^2}\vec{r}_{ab} + \alpha m_b\left(\frac{\vec{p}_b}{m_b} - \frac{\vec{p}_a}{m_a}\right)\right]\right\}\\
&= -\frac{1}{2c^2M_o}\sum_a\left\{m_av_a^2\vec{v}_a + \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}}\left[\vec{v}_b + \frac{(\vec{v}_b\cdot\vec{r}_{ab})}{r_{ab}^2}\vec{r}_{ab}\right]\right\}\\
&= -\frac{1}{2c^2M_o}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_a m_a\left(\vec{r}_a\cdot\vec{v}_a\right)\vec{v}_a\right]\\
&\quad + \frac{1}{2c^2M_o}\sum_a\left\{m_a\left[\left(\vec{r}_a\cdot\frac{\mathrm{d}\vec{v}_a}{\mathrm{d}t}\right)\vec{v}_a + (\vec{r}_a\cdot\vec{v}_a)\frac{\mathrm{d}\vec{v}_a}{\mathrm{d}t}\right] - \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}}\left[\vec{v}_b + \frac{(\vec{v}_b\cdot\vec{r}_{ab})}{r_{ab}^2}\vec{r}_{ab}\right]\right\}.
\end{aligned} \tag{7}$$

We note that the $\alpha$-containing term in the second row, which indicates the difference between the gravitational and electromagnetic interactions, disappears in the total sum, and in the following both interactions can be treated with the same expressions, as is the case in the Newtonian limit.

Since the last row in (7) is already of order $1/c^2$, it suffices to substitute in it the Newtonian equations of motion. With the single particle equation

$$m_a\frac{\mathrm{d}\vec{v}_a}{\mathrm{d}t} = \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}\vec{r}_{ab} \tag{8}$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[2c^2M_o\vec{R}_o + \sum_a m_a(\vec{r}_a\cdot\vec{v}_a)\vec{v}_a\right]\\
&\qquad = \sum_a\sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}[(\vec{r}_a\cdot\vec{r}_{ab})\vec{v}_a + (\vec{r}_a\cdot\vec{v}_a)\vec{r}_{ab} - r_{ab}^2\vec{v}_b - (\vec{v}_b\cdot\vec{r}_{ab})\vec{r}_{ab}]\\
&\qquad = \sum_a\sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}[(\vec{r}_b\cdot\vec{r}_{ab})\vec{v}_a + (\vec{r}_b\cdot\vec{v}_a)\vec{r}_{ab}].
\end{aligned} \tag{9}$$

Therefore, if there exists a vector $\vec{W}$ such that

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \sum_a\sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}[(\vec{r}_b\cdot\vec{r}_{ab})\vec{v}_a + (\vec{r}_b\cdot\vec{v}_a)\vec{r}_{ab}], \tag{10}$$

then equation (9) may be integrated:

$$\vec{R}_o = \vec{X}_o - \frac{1}{2M_oc^2}\left[\sum_a m_a(\vec{r}_a\cdot\vec{v}_a)\vec{v}_a - \vec{W}\right], \tag{11}$$

with $\vec{X}_o$ being an arbitrary integration constant, so that the Lorentz boost (3) becomes

$$\begin{aligned}\vec{N} &= M\vec{X}_o + \frac{1}{2c^2}\left\{\sum_a m_a\left[\vec{v}_a^2\vec{r}_a - (\vec{r}_a\cdot\vec{v}_a)\vec{v}_a\right] + \sum_{(a,b)}\frac{\kappa_{ab}}{r_{ab}}(\vec{r}_a+\vec{r}_b) + \vec{W}\right\}\\ &= M\vec{X}_o + \frac{1}{2c^2}\left[\sum_a m_a\vec{v}_a\times(\vec{r}_a\times\vec{v}_a) + \sum_{(a,b)}\frac{\kappa_{ab}}{r_{ab}}(\vec{r}_a+\vec{r}_b) + \vec{W}\right].\end{aligned} \tag{12}$$

Defining the vector

$$\vec{A} \equiv \sum_a m_a\vec{v}_a\times(\vec{r}_a\times\vec{v}_a) + \sum_{(a,b)}\frac{\kappa_{ab}}{r_{ab}}(\vec{r}_a+\vec{r}_b) + \vec{W} \tag{13}$$

the boost may finally be brought to the generic form

$$\vec{N} = M\vec{X}_o + \frac{1}{2c^2}\vec{A}. \tag{14}$$

Since $\vec{N}$, $M$ and $\vec{X}_o$ are all constants, the vector $\vec{A}$ is clearly an integral of the motion in the non-relativistic limit. Its constancy may of course be verified now in a straightforward computation using the Newtonian equations of motion (8) and equation (10). That $\vec{A}$ is a vector of the Laplace–Runge–Lenz type is evident from the fact that its kinetic part is composed of terms which are of second order in the velocities and of first order in the coordinates, related via the cross product.

Since $\vec{A}$ is an internal vector it follows from equation (14) that all the translational properties of $\vec{N}$ are contained in $\vec{X}_o$. As is discussed in [14] for 2-body systems, it is $\vec{X}_o$—the arbitrary constant of integration—that should be identified as the post-Newtonian centre-of-mass, not the so-called centre-of-inertia

$$\vec{R}_I \equiv \frac{\vec{N}}{M}. \tag{15}$$

The centre-of-inertia is therefore shifted from the centre-of-mass by a vector which is determined by $\vec{A}$:

$$\vec{R}_I = \vec{X}_o + \frac{1}{2M_oc^2}\vec{A}. \tag{16}$$

$M$ could be replaced by $M_o$ in the second term because it is already of order $1/c^2$. $\vec{X}_o$ and $\vec{R}_I$ become identical only in the non-relativistic limit.

## 3. Integrability of $\vec{W}$

Our ultimate goal—finding a new, generally valid extra integral of motion in many-body systems—would be achieved if we could integrate equation (10) for arbitrary systems. The integrability considered here is regarding the possibility of $\vec{W}$ being a relatively simple, analytic function of the dynamic variables of the particles; as a general function of coordinates and momenta, equation (10) is expected to be always integrable, though not necessarily in terms of analytic functions.

Since the rhs of equation (10) is linear in the velocities, its integral (in this simple sense) is expected to be in terms of the coordinates alone, $\vec{W} = \vec{W}(\{\vec{r}_a\})$. Although such an integral does not exist in the general case, there exist certain families of solutions, relatively simple but nevertheless non-trivial, for which integration is possible. In particular, there exist those for which the rhs of equation (10) vanishes. Then we may assume that $\vec{W} = 0$ without loss of generality, so that $\vec{A}$ reduces to

$$\vec{A} = \sum_a m_a \vec{v}_a \times (\vec{r}_a \times \vec{v}_a) + \sum_{(a,b)} \frac{\kappa_{ab}}{r_{ab}} (\vec{r}_a + \vec{r}_b), \tag{17}$$

still an integral of motion. Let us consider these solutions.

Using the identity

$$\vec{r}_a = \sum_b \frac{m_b \vec{r}_{ab}}{M_o} \tag{18}$$

which is easily verified from equation (2), and defining the 3-body vectors

$$\vec{w}_{abc} \equiv \frac{m_c \kappa_{ab}}{r_{ab}^3} \vec{r}_{ab} + \frac{m_a \kappa_{bc}}{r_{bc}^3} \vec{r}_{bc} + \frac{m_b \kappa_{ac}}{r_{ac}^3} \vec{r}_{ca} \tag{19}$$

it is possible to transform the rhs of equation (10) into a sum over triplets:

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \frac{1}{M_o} \sum_a \sum_{(b,c)\neq a} [(\vec{r}_{bc} \cdot \vec{w}_{abc}) \vec{v}_a + (\vec{r}_{bc} \cdot \vec{v}_a) \vec{w}_{abc}], \tag{20}$$

where the sum over $(b, c)$ is over all pairs that do not include the $a$th particle. The vectors $\vec{w}_{abc}$ are symmetric under even permutations of the indices $abc$ and anti-symmetric under odd permutations. Evidently, if all the $\vec{w}_{abc}$ vanish, then $\vec{W} = 0$ and equation (17) is obtained.

The simplest case is that of 2-body systems. Dahl's result [13] applies to all 2-body systems, so we expect that equation (10) be integrable for all such systems. Indeed, since the vectors $\vec{w}_{abc}$ correspond to triplets (three different particles) they must vanish identically for 2-body systems. Thus, by the previous argument, $\vec{W} = 0$. The vector $\vec{A}$ (17) is then an integral of the motion which is proportional to the Runge–Lenz vector (1),

$$\vec{A} = \frac{m_2 - m_1}{m_1 + m_2} \left[ \mu \vec{v} \times (\vec{r} \times \vec{v}) + \frac{\kappa}{r} \vec{r} \right] = \frac{m_2 - m_1}{m_1 + m_2} \vec{K}, \tag{21}$$

which is Dahl's result [13]. Clearly, $\vec{A}$ vanishes when the masses are equal. This is to be expected, since in this case, by symmetry, the centre-of-mass must be identical with the geometrical centre, and no shift can occur between the centre-of-mass and the centre-of-inertia.

In many-body systems this simplicity is lost, and the integrability of equation (10) is not ensured. We turn therefore to study the conditions under which $\vec{W}$ is integrable.

## 4. Many-body collinear configurations

As a first step, let us consider an $N$-body system ($N$ arbitrary) in which all the particles are co-aligned, situated at any moment on one straight line (the orientation of the line may change in time). Then it can be shown [7–10] that they must be situated with constant ratios between their distances to the centre-of-mass: $\vec{r}_a(t) = \rho_a \vec{r}(t)$ for all $a$, where the $\rho_a$'s are constant and $\vec{r}(t)$ is a common vector. A theorem by Moulton [16] then states that for any ordering of the particles (according to their masses) there is a unique set of coefficients $\{\rho_a\}$ for which the equations of motion are satisfied. Then also $\vec{r}_{ab}(t) = \rho_{ab} \vec{r}(t)$ with $\rho_{ab} = \rho_a - \rho_b$, and equation (10) yields

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \sum_a \sum_{b\neq a} \frac{\kappa_{ab} \rho_a \rho_b \rho_{ab}}{|\rho_{ab}|^3 r^3} [r^2 \dot{\vec{r}} + (r\dot{r})\vec{r}] = 0. \tag{22}$$

The sum vanishes for each pair separately due to the anti-symmetry of $\rho_{ab}$. Then again $\vec{W} = 0$ and the vector $\vec{A}$ (equation (17)) is indeed an integral of the motion.

For the explicit computation of $\vec{A}$ we use the fact that as a consequence of the particles' equations of motion (8) the common vector $\vec{r}(t)$ also satisfies an equation of the Kepler/Coulomb type:

$$\frac{d^2\vec{r}}{dt^2} = \frac{\kappa_o}{r^3}\vec{r}, \tag{23}$$

where $\kappa_o$ is a constant that satisfies for all $a$:

$$\sum_{b\neq a} \frac{\kappa_{ab}}{|\rho_{ab}|^3}\rho_{ab} = m_a\rho_a\kappa_o. \tag{24}$$

From equation (24) we obtain further the relation

$$\sum_{(a,b)} \frac{\kappa_{ab}}{|\rho_{ab}|}(\rho_a + \rho_b) = \sum_{(a,b)} \frac{\kappa_{ab}}{|\rho_{ab}|^3}\rho_{ab}\left(\rho_a^2 - \rho_b^2\right) = \sum_a \sum_{b\neq a} \frac{\kappa_{ab}}{|\rho_{ab}|^3}\rho_{ab}\rho_a^2$$
$$= \kappa_o \sum_a m_a\rho_a^3. \tag{25}$$

As a solution of equation (23) the common vector $\vec{r}(t)$ traverses a conic-section orbit. Substituting $\vec{r}_a(t) = \rho_a\vec{r}(t)$ into equation (17) and using equation (25) we obtain

$$\vec{A} = \left(\sum_a m_a\rho_a^3\right)\vec{K}_o, \tag{26}$$

where $\vec{K}_o$ is the Runge–Lenz vector associated with the $\vec{r}(t)$-orbit:

$$\vec{K}_o = \dot{\vec{r}} \times (\vec{r} \times \dot{\vec{r}}) + \frac{\kappa_o}{r}\vec{r}. \tag{27}$$

This result is clearly independent of the particular time dependence of the common vector $\vec{r}(t)$, except for the constant vector $\vec{K}_o$.

## 5. 3-body systems

The collinear configurations are part of what are known in celestial mechanics as *central configurations* [7–10], referring to the configurations in which all the particles' accelerations are radial with respect to the centre-of-mass by the same ratio

$$\frac{d\vec{v}_a}{dt} = \Gamma\vec{r}_a, \tag{28}$$

where $\Gamma$ is a scalar (not necessarily constant) common to all the particles.

From their definition, all the vectors $\{\vec{r}_a\}$ are linearly dependent via the relation

$$\sum_a m_a\vec{r}_a = 0. \tag{29}$$

If they are otherwise linearly independent (which is the case for three bodies in a plane or four bodies in 3D), then the central configuration condition (28) necessarily implies that $\vec{w}_{abc} = 0$: combining equations (8) and (28), the latter may be written as

$$\left(m_a\Gamma - \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}\right)\vec{r}_a + \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3}\vec{r}_b = 0. \tag{30}$$

Since relations (29) and (30) must be identical up to a multiplying factor, let us denote this factor $\lambda_a$, so that the correspondence between the relations leads, for any $a$, to

$$m_a\Gamma - \sum_{b\neq a}\frac{\kappa_{ab}}{r_{ab}^3} = m_a\lambda_a \tag{31a}$$

$$\frac{\kappa_{ab}}{r_{ab}^3} = m_b \lambda_a \qquad (\forall b \neq a). \tag{31b}$$

Substituting equation (31*b*) into (31*a*) leads to $\lambda_a = m_a \Gamma / M_o$, and again in (31*b*) implies

$$\frac{\kappa_{ab}}{r_{ab}^3} = \frac{m_a m_b \Gamma}{M_o} \qquad (\forall a, b).$$

Hence for any triplet the vectors $\vec{w}_{abc}$ become

$$\vec{w}_{abc} = \frac{m_a m_b m_c \Gamma}{M_o} (\vec{r}_{ab} + \vec{r}_{bc} + \vec{r}_{ca})$$

and they do indeed vanish due to the triangle condition

$$\vec{r}_{ab} + \vec{r}_{bc} + \vec{r}_{ca} = 0. \tag{32}$$

In the case of three bodies this configuration is known in celestial mechanics as the *Lagrangian triangular configuration* [8]. It is the simplest non-collinear central configuration, and is discussed in detail in the next section, culminating in the computation of the integral vector $\vec{A}$ corresponding to this configuration.

The foregoing results together with those of the previous section suggest that $\vec{W}$ vanishes for central configurations. We now show, for a 3-body system, that these are the only cases of integrability.

Let us write equation (20) in differential form for a 3-body system:

$$\begin{aligned} \mathrm{d}\vec{W} = \frac{1}{M_o}[(\vec{r}_{12} \cdot \vec{w})\,\mathrm{d}\vec{r}_3 + (\vec{r}_{23} \cdot \vec{w})\,\mathrm{d}\vec{r}_1 + (\vec{r}_{31} \cdot \vec{w})\,\mathrm{d}\vec{r}_2 \\ + (\vec{r}_{12} \cdot \mathrm{d}\vec{r}_3 + \vec{r}_{23} \cdot \mathrm{d}\vec{r}_1 + \vec{r}_{31} \cdot \mathrm{d}\vec{r}_2)\vec{w}] = 0, \end{aligned} \tag{33}$$

with $\vec{w} = \vec{w}_{123}$. It is convenient to express the particles' coordinates in terms of Jacobi variables $\vec{y}_1$ and $\vec{y}_2$:

$$\begin{aligned} \vec{r}_1 = \frac{m_2 \vec{y}_1}{M_2} + \frac{m_3 \vec{y}_2}{M_o}, \qquad \vec{r}_2 = \frac{m_3 \vec{y}_2}{M_o} - \frac{m_1 \vec{y}_1}{M_2}, \qquad \vec{r}_3 = -\frac{M_2 \vec{y}_2}{M_o}, \\ \vec{r}_{12} = \vec{y}_1, \qquad \vec{r}_{23} = \vec{y}_2 - \frac{m_1 \vec{y}_1}{M_2}, \qquad \vec{r}_{13} = \frac{m_2 \vec{y}_1}{M_2} + \vec{y}_2, \end{aligned} \tag{34}$$

with $M_2 = m_1 + m_2$, and we obtain from equation (33)

$$\begin{aligned} \mathrm{d}\vec{W} &= \frac{1}{M_o}[(\vec{y}_2 \cdot \vec{w})\,\mathrm{d}\vec{y}_1 - (\vec{y}_1 \cdot \vec{w})\,\mathrm{d}\vec{y}_2 + (\vec{y}_2 \cdot \mathrm{d}\vec{y}_1 - \vec{y}_1 \cdot \mathrm{d}\vec{y}_2)\vec{w}] \\ &= \frac{1}{M_o}\mathrm{d}[(\vec{y}_2 \cdot \vec{w})\vec{y}_1 - (\vec{y}_1 \cdot \vec{w})\vec{y}_2] \\ &\quad + \frac{1}{M_o}[(\vec{y}_2 \cdot \mathrm{d}\vec{y}_1 - \vec{y}_1 \cdot \mathrm{d}\vec{y}_2)\vec{w} + \mathrm{d}(\vec{y}_1 \cdot \vec{w})\vec{y}_2 - \mathrm{d}(\vec{y}_2 \cdot \vec{w})\vec{y}_1]. \end{aligned} \tag{35}$$

Integrability of equation (35) implies that the last row there must vanish, so that

$$(\vec{y}_2 \cdot \mathrm{d}\vec{y}_1 - \vec{y}_1 \cdot \mathrm{d}\vec{y}_2)\,\vec{w} = \mathrm{d}(\vec{y}_2 \cdot \vec{w})\vec{y}_1 - \mathrm{d}(\vec{y}_1 \cdot \vec{w})\vec{y}_2. \tag{36}$$

$\vec{w}$ is a linear combination of $\vec{y}_1$ and $\vec{y}_2$, which we write as $\vec{w} = w_1 \vec{y}_1 + w_2 \vec{y}_2$. Then, due to the independence of $\vec{y}_1$ and $\vec{y}_2$, we obtain from equation (36) two scalar conditions:

$$\begin{aligned} (\vec{y}_1 \times \vec{y}_2)^2\,\mathrm{d}w_1 &= 2(\vec{y}_1 \cdot \vec{y}_2)\vec{w} \cdot \mathrm{d}\vec{y}_2 - 2\vec{y}_2^2 \vec{w} \cdot \mathrm{d}\vec{y}_1 \\ &= (\vec{y}_1 \cdot \vec{y}_2) w_2\,\mathrm{d}\vec{y}_2^2 - \vec{y}_2^2 w_1\,\mathrm{d}\vec{y}_1^2 + 2(\vec{y}_1 \cdot \vec{y}_2) w_1 \vec{y}_1 \cdot \mathrm{d}\vec{y}_2 - 2\vec{y}_2^2 w_2 \vec{y}_2 \cdot \mathrm{d}\vec{y}_1 \end{aligned} \tag{37}$$

$$\begin{aligned} (\vec{y}_1 \times \vec{y}_2)^2\,\mathrm{d}w_2 &= 2(\vec{y}_1 \cdot \vec{y}_2)\vec{w} \cdot \mathrm{d}\vec{y}_1 - 2\vec{y}_1^2 \vec{w} \cdot \mathrm{d}\vec{y}_2 \\ &= (\vec{y}_1 \cdot \vec{y}_2) w_1\,\mathrm{d}\vec{y}_1^2 - \vec{y}_1^2 w_2\,\mathrm{d}\vec{y}_2^2 + 2(\vec{y}_1 \cdot \vec{y}_2) w_2 \vec{y}_2 \cdot \mathrm{d}\vec{y}_1 - 2\vec{y}_1^2 w_1 \vec{y}_1 \cdot \mathrm{d}\vec{y}_2. \end{aligned} \tag{38}$$

In planar configurations $\vec{y}_1$ and $\vec{y}_2$ are linearly independent and $\vec{y}_1 \times \vec{y}_2 \neq 0$. Since $w_1$ and $w_2$ are both scalar functions of $\vec{y}_1$ and $\vec{y}_2$ via the scalar products $(\vec{y}_1 \cdot \vec{y}_2)$, $\vec{y}_1^2$ and $\vec{y}_2^2$, it then follows that the coefficients of $\vec{y}_2 \cdot d\vec{y}_1$ and $\vec{y}_1 \cdot d\vec{y}_2$ must be equal, so that

$$w_1\vec{y}_1^2 + w_2(\vec{y}_1 \cdot \vec{y}_2) = \vec{w} \cdot \vec{y}_1 = 0 = w_1 (\vec{y}_1 \cdot \vec{y}_2) + w_2\vec{y}_2^2 = \vec{w} \cdot \vec{y}_2 = 0 \tag{39}$$

and the only possible solution is when $\vec{w} = 0$. In collinear configurations, we simply use the one-dimensional notation $y_1$ and $y_2$ instead of the vectors and obtain, say from equation (37),

$$y_1 y_2 w \, dy_2 - y_2^2 w \, dy_1 = y_2 w(y_1 \, dy_2 - y_2 \, dy_1) = 0, \tag{40}$$

with a similar relation from equation (38). Thus, either $w = 0$ or $y_1$ and $y_2$ maintain the constant ratio $y_1/y_2$, verifying that the configurations discussed above are indeed the only ones for which $\vec{W}$ is integrable in 3-body systems.

The vanishing of $\vec{W}$ and the consequent integrability of $\vec{A}$ for the general collinear configurations or triangular 3-body central configurations strongly suggest that this is the case for all central configurations for all $N$. The verification of this conjecture is still open.

## 6. Triangular central configurations

Let us consider in the following the central triangular configurations, as another example for non-trivial configurations for which $\vec{A}$ is an integral of the motion. Appreciating that the general reader is not familiar with these configurations, let us start with a review of their properties in a way that is also appropriate for our purposes. From the single particle equation of motion (8), we obtain the equation of motion for the relative coordinates

$$m_a m_b \frac{d\vec{v}_{ab}}{dt} = M_o \frac{\kappa_{ab}}{r_{ab}^3}\vec{r}_{ab} - \sum_{c\neq a,b} \vec{w}_{abc}. \tag{41}$$

Assuming that all the $\vec{w}_{abc}$ vanish, the equations of motion (41) of the various relative coordinates separate and become independent:

$$\frac{d\vec{v}_{ab}}{dt} = \frac{M_o\kappa_{ab}}{m_a m_b}\frac{\vec{r}_{ab}}{r_{ab}^3}. \tag{42}$$

These are Kepler/Coulomb equations of motion, with the constants that are associated with the solution of each equation:

$$\eta_{ab} = \frac{1}{2}v_{ab}^2 + \frac{M_o\kappa_{ab}}{m_a m_b}\frac{1}{r_{ab}} \tag{43}$$

$$\vec{\lambda}_{ab} = \vec{r}_{ab} \times \vec{v}_{ab} \tag{44}$$

$$\vec{K}_{ab} = \vec{v}_{ab} \times (\vec{r}_{ab} \times \vec{v}_{ab}) + \frac{M_o\kappa_{ab}}{m_a m_b}\frac{\vec{r}_{ab}}{r_{ab}} \tag{45}$$

with the well-known relations between these constants:

$$K_{ab}^2 = 2\eta_{ab}\lambda_{ab}^2 + \left(\frac{M_o\kappa_{ab}}{m_a m_b}\right)^2. \tag{46}$$

The solutions of equation (42) are conic sections with eccentricity

$$\varepsilon = \frac{m_a m_b|\vec{K}_{ab}|}{M_o|\kappa_{ab}|} = \sqrt{1 + 2\eta_{ab}\lambda_{ab}^2\left(\frac{m_a m_b}{M_o\kappa_{ab}}\right)^2} \tag{47}$$

and $\vec{K}_{ab}$ is the constant Runge–Lenz vector associated with these solutions.

Any three particles $a, b, c$ form a triangle. The vanishing $\vec{w}_{abc}$ condition now implies

$$\frac{\kappa_{ab}}{m_a m_b r_{ab}^3}\vec{r}_{ab} + \frac{\kappa_{bc}}{m_a m_c r_{bc}^3}\vec{r}_{bc} + \frac{\kappa_{ac}}{m_b m_c r_{ac}^3}\vec{r}_{ca} = 0. \tag{48}$$

The three vectors $\vec{r}_{ab}, \vec{r}_{bc}, \vec{r}_{ca}$ may be either collinear or not, so let us consider the triangular case. Equation (48), together with the triangle condition (32), then implies the condition

$$\frac{\kappa_{ab}}{m_a m_b r_{ab}^3} = \frac{\kappa_{bc}}{m_a m_c r_{bc}^3} = \frac{\kappa_{ac}}{m_b m_c r_{ac}^3}. \tag{49}$$

In the case of gravitational systems

$$\frac{\kappa_{ab}}{m_a m_b} = -G$$

so that all the relative distances must be equal at all times, $r_{ab}(t) = r(t)\,\forall a, b$, and the particles form an equilateral triangle. In the case of electromagnetic systems all the $\kappa_{ab}$ must have the same sign, hence all the charges having the same sign and all the $\kappa_{ab}$ strictly positive. All the relative distances scale to a common temporal dependence:

$$r_{ab}(t) = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{1}{3}} a(t) \tag{50}$$

(we keep the absolute sign for $|\kappa_{ab}|$ so that equation (50) and the following discussion will be appropriate for both electromagnetic and gravitational systems). From equations (43)–(45) it follows that $\eta_{ab}$ and $\vec{\lambda}_{ab}$ scale as $\left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{2/3}$ and $\vec{K}_{ab}$ scales as $\left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)$. Thus, there are constants $\eta$, $\lambda$ and $K$ such that for all $a, b$

$$\eta_{ab} = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} \eta \tag{51}$$

$$|\vec{\lambda}_{ab}| = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} \lambda \tag{52}$$

$$|\vec{K}_{ab}| = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right) K. \tag{53}$$

From equation (47) it follows that

$$\varepsilon = K = \sqrt{2\eta\lambda^2 + 1} \tag{54}$$

and $\eta$ is determined by the total internal energy:

$$E = \sum_{(a,b)} \frac{m_a m_b}{M_o}\eta_{ab} = \left[\sum_{(a,b)} \frac{m_a m_b}{M_o}\left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}}\right]\eta. \tag{55}$$

The vectors $\vec{r}_{ab}$ and $\vec{v}_{ab}$ are confined to the plane perpendicular to $\vec{\lambda}_{ab}$. This plane is spanned by $\vec{K}_{ab}$ and $\vec{\lambda}_{ab} \times \vec{K}_{ab}$, and it may be shown that the triangle condition (32) together with the corresponding one for the velocities

$$\vec{v}_{ab} + \vec{v}_{bc} + \vec{v}_{ca} = 0 \tag{56}$$

is embodied in the following relations:

$$\begin{gathered}\left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{2}{3}} \vec{K}_{ab} + \left(\frac{M_o\,|\kappa_{bc}|}{m_b m_c}\right)^{-\frac{2}{3}} \vec{K}_{bc} + \left(\frac{M_o\,|\kappa_{ac}|}{m_a m_c}\right)^{-\frac{2}{3}} \vec{K}_{ca} = 0 \\ \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{4}{3}} \vec{\lambda}_{ab} \times \vec{K}_{ab} + \left(\frac{M_o|\kappa_{bc}|}{m_b m_c}\right)^{-\frac{4}{3}} \vec{\lambda}_{bc} \times \vec{K}_{bc} + \left(\frac{M_o\,|\kappa_{ac}|}{m_a m_c}\right)^{-\frac{4}{3}} \vec{\lambda}_{ca} \times \vec{K}_{ca} = 0.\end{gathered} \tag{57}$$

Then it can be shown that the two conditions in equation (57), together with the orthogonality relations $\vec{\lambda}_{ab} \cdot \vec{K}_{ab} = 0$ for all $a, b$, can co-exist only if all the vectors $\vec{\lambda}_{ab}$ are parallel. Thus, all the motion is confined to the plane formed by the three particles, known as the 'invariant plane' in celestial mechanics [8]. If there are more than three particles, then all the triangles formed by all possible triplets should all be in the same plane, but that would be impossible to reconcile with the relations required by equation (50). Consequently, these triangular central configurations are possible only in 3-body systems (4-body solutions are possible only if all $\vec{\lambda}_{ab} = 0$, which are tetrahedral configurations uniformly contracting or expanding, but not rotating). Then, in any such system, there exists a vector $\vec{\lambda}$ such that

$$\vec{\lambda}_{ab} = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} \vec{\lambda}, \qquad (ab) = (12), (23), (31) \tag{58}$$

so that the total internal angular momentum is

$$\vec{\ell} = \sum_{(a,b)} \frac{m_a m_b}{M_o} \vec{\lambda}_{ab} = \left[\sum_{(a,b)} \frac{m_a m_b}{M_o} \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}}\right] \vec{\lambda}. \tag{59}$$

Finally, for the computation of $\vec{A}$ it suffices, due to its constancy, to be done at one particular point. The most convenient one is the point of the maximal approach (minimal distance) of the particles. Since all the $\vec{r}_{ab}$ follow, by equation (50), the same temporal pattern, let $t = 0$ denote the time of the minimal distances between all the particles. At the extrema of the orbit $\vec{r}_{ab} \cdot \vec{v}_{ab} = 0$, and the minimum of the distance function $a(t)$ is

$$a_{\min} = \frac{\varepsilon \pm 1}{2\eta} = \frac{\lambda^2}{\varepsilon \mp 1}, \tag{60}$$

where $\pm = \text{sign}\,(\kappa_{ab})$. Then the relative vectors and the relative velocities are

$$\begin{aligned} \vec{r}_{ab}(0) &= \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{2}{3}} \frac{\lambda^2 \pm a_{\min}}{\varepsilon^2} \vec{K}_{ab} = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{2}{3}} \frac{a_{\min}}{\varepsilon} \vec{K}_{ab} \\ \vec{v}_{ab}(0) &= \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{2}{3}} \left(2\eta \mp \frac{1}{a_{\min}}\right) \frac{\vec{\lambda} \times \vec{K}_{ab}}{\varepsilon^2} = \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{-\frac{2}{3}} \frac{\vec{\lambda} \times \vec{K}_{ab}}{\varepsilon a_{\min}}. \end{aligned} \tag{61}$$

These relations may be combined, using equation (60), to yield

$$\vec{v}_{ab}(0) \times \vec{\lambda} = \frac{\lambda^2}{a_{\min}^2} \vec{r}_{ab}(0) = \frac{\varepsilon \mp 1}{a_{\min}} \vec{r}_{ab}(0).$$

Then, using equation (18) and the corresponding identity for the velocities

$$\vec{v}_a = \sum_b \frac{m_b \vec{v}_{ab}}{M_o}, \tag{62}$$

a similar relation is obtained for the single particles:

$$\vec{v}_a(0) \times \vec{\lambda} = \frac{\lambda^2}{a_{\min}^2} \vec{r}_a(0) = \frac{\varepsilon \mp 1}{a_{\min}} \vec{r}_a(0).$$

For the computation of $\vec{A}$ it is convenient to transform the first sum in equation (13) into summation over pairs of particles. Using identities (18) and (62) we obtain

$$\vec{A} \equiv \sum_{(a,b)} \left[\frac{m_a m_b}{M_o} (\vec{v}_a + \vec{v}_b) \times (\vec{r}_{ab} \times \vec{v}_{ab}) + \frac{\kappa_{ab}}{r_{ab}} (\vec{r}_a + \vec{r}_b)\right]. \tag{63}$$

Since $\vec{A}$ refers to non-relativistic dynamics, $\vec{R}_o$ may be replaced in the computation of $\vec{r}_a$ by its non-relativistic limit $\vec{X}_o$. Thus we finally obtain for the vector $\vec{A}$:

$$\begin{aligned}\vec{A} &= \sum_{(a,b)} \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} \frac{m_a m_b}{M_o} \left\{[\vec{v}_a(0) + \vec{v}_b(0)] \times \vec{\lambda} \pm \frac{1}{a_{\min}}[\vec{r}_a(0) + \vec{r}_b(0)]\right\} \\ &= \sum_{(a,b)} \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} \frac{m_a m_b}{M_o} \left(\frac{\varepsilon \mp 1}{a_{\min}} \pm \frac{1}{a_{\min}}\right)[\vec{r}_a(0) + \vec{r}_b(0)] \\ &= \frac{\varepsilon}{M_o a_{\min}} \sum_{(a,b)} \left(\frac{M_o|\kappa_{ab}|}{m_a m_b}\right)^{\frac{2}{3}} m_a m_b\,[\vec{r}_a(0) + \vec{r}_b(0)].\end{aligned} \tag{64}$$

In the gravitational case, or for electromagnetic systems with all the particles having identical charge-to-mass ratios $(e_a/m_a)$, equation (64) may be further simplified using the identity

$$\sum_{(a,b)} m_a m_b(\vec{r}_a + \vec{r}_b) = -\sum_a m_a^2 \vec{r}_a \tag{65}$$

and we obtain

$$\vec{A} = -\frac{\varepsilon}{M_o a_{\min}} \left(\frac{M_o\,|\kappa_{12}|}{m_1 m_2}\right)^{\frac{2}{3}} \sum_a m_a^2 \vec{r}_a(0). \tag{66}$$

## 7. Concluding remarks

A new constant of motion—the vector $\vec{A}$ defined in equation (13)—was found in many-body electrical or gravitational systems. For central configurations, it reduces to an integral of motion—an algebraically simple vector observable—as given by equation (17). That it is new, not just a function of the classical integrals (energy, linear and angular momentum), is verified from the fact that it is of the Runge–Lenz type.

Christian Marchal, in his book *The Three-Body Problem* [8] which is one of the relatively recent publications on the subject, reviews briefly the unsuccessful history of searching for integrals of motion other than the global ones in many-body gravitational systems. He concludes with the words (p 25):

*Thus the conjectural absence of new non-classical integrals means absence of integrals that would be*

*(A) independent of time,*
*(B) continuous in terms of the present state,*
*(C) non-transitory,*
*(D) isolating,*
*(E) useful even for bounded and oscillatory orbits.*

It is evident that our $\vec{A}$ stands in contrast to this statement.

It has been verified that $\vec{A}$ is indeed an integral of motion for collinear central configurations for arbitrary number of particles $N$, and for the central triangular configurations of three bodies. It has also been shown that for three bodies these are the only configurations for which $\vec{A}$ is an integral. It still remains to be verified that (as may be conjectured from the foregoing results) $\vec{A}$ is an integral also for non-collinear central configurations for more than three bodies. It also remains to see whether, and for what configurations, $\vec{A}$ is an integral in restricted 3-body systems.

In 2-body systems, knowledge of the Runge–Lenz vector amounts to having a full solution for the configuration of the system. In such systems, the internal dynamics is governed by the symmetry generated together by the internal angular momentum and the Runge–Lenz vector. Another direction for inquiry is therefore what information is contained in $\vec{A}$ regarding the configuration, and what are the symmetry properties associated with it.

The only non-trivial explicit solutions which are so far known for gravitational many-body systems are central configurations [8, 9], especially collinear ones. Non-collinear solutions are apparently known only for three and four bodies. Consequently, it is suggested that the vector $\vec{A}$ may be of much help in the analysis of such configurations in many-body systems, and also in the analysis of more general systems.

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